AN EXAMPLE OF A DOMINANCE APPROACH TO RATIONAL EXPECTATIONS *

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Dominance and independence produce an alternative approach to rational expectations in a simple stationary overlapping-generations model. Dominance and independence delimit rational behavior, but predict a range of rational choice rather than a single rational choice. In particular, a coherent interpretation of multiple rational expectations equilibria is provided.

1. Introduction

This letter suggests, and describes through an example, a dominance approach to rational expectations as an alternative to the perfect foresight fixed point for closing a model. Dominance criteria have a long history in game theory [see, e.g., Harsanyi (1977, p. 107)]. It also has long been recognized that dominance is not a universally applicable model of behavior [see e.g., Harsanyi (1977, p. 122)]. However, it is argued in Bryant (1982) that under 'independence' rational individuals base their decisions upon standard game theoretic vector, or pointwise, dominance. Moreover, 'independence' holds if the problems of the relevant other economic agents do not depend upon the decisions of the individual agent in question. To isolate on expectations, this letter treats a simple dynamic model exhibiting such independence. The contribution of this example is to show that the standard game theoretic contraction by dominance procedure can be fruitfully applied to the problem of expectation formation in a dynamic setting.

* Comments appreciated.
Recently an alternative to the standard game theoretic dominance procedure has been suggested by Bernheim (n.d.) and Pearce (1983): 'rationalizability'. They suggest that only best-reply [Harsanyi (1977, p. 116)] strategies be considered. This suggestion seems to be a bit at variance with the spirit of game theory. Bernheim and Pearce assume that individuals use probability measures on other individuals' strategies, although Bernheim's and Pearce's procedure does not prescribe any particular probability measures (except when it is equivalent to Nash equilibrium). They rule out a strategy which is not a best-reply to any set of mixed strategies of other individuals, even if the ruled out strategy is better than each particular best-reply strategy for some set of strategies of other individuals. In any case, there does not seem to be any compelling reason to abandon the standard game theoretic procedure in favor of Bernheim's and Pearce's untried approach in a study of expectation formation in a dynamic setting.

To more clearly expostive dominance and independence some formalization is useful. Suppose the agent in question must choose a value for \( w \), (an)other agent(s) must choose a value for \( w' \), and the agent in question then gets a payoff of \( V(w, w') \). Suppose the agent in question realizes that \( w' \) belongs to some set \( D \), but does not know the particular value or have a probability measure on \( D \). If the agent in question assumes the choice of \( w' \) is independent of his/her choice of \( w \), then it is argued in Bryant (1982) that rationally that agent should use the vector dominance criterion. Specifically that agent should not choose \( w = \bar{w} \) if he/she could choose \( w = \bar{w} \) where \( V(\bar{w}, w') < V(\bar{w}, w') \) for all \( w' \) in \( D \). Agents do not choose inadmissable strategies. It is the assumption of this behavior which we use to operationalize rational expectations. Typically the resulting procedure predicts that there is a range of rational behavior, rather than a single rational outcome.

2. The example

Our example of the use of dominance and independence to delimit rational behavior is a simple stationary overlapping-generations model. For expectations to be an interesting concept we need a model in which futures markets do not operate, which is a characteristic of overlapping-generations models. The simple structure and stationarity are for clarity. In our example we isolate on expectations.
Now let us turn to the details of the structure of the example. There is a countable infinity of individuals indexed \(-1, 0, 1, \ldots\). Time is discrete without beginning or end, and indexed \(-1, 0, 1, \ldots\). Individual \(i\) lives in period \(i\) and in period \(i+1\). In period \(i+1\) individual \(i\) consumes a single non-storable consumption good, and cares for nothing else. In period \(i\) individual \(i\) chooses \(w_i\) and receives in period \(i+1\) the utility from consuming of \(V(w_i, w_{i+1})\). Notice in particular that the payoff to individual \(i+1\) does not depend upon \(w_i\) and therefore, as argued in Bryant (1982), individual \(i\) may treat \(w_{i+1}\) as independent of \(w_i\). This we assume. Notice also that individual \(i\) cannot pay individual \(i+1\) for his/her decision as \(i\) has nothing anyone else wants. Futures markets do not operate.

We now proceed more formally to show that dominance and independence produce an approach to rational expectations in our example.

We assume \(V(w, w')\) is strictly concave, and therefore continuous. We assume that \(w, w' \in I \times I\), where \(I = [0, 1]\). Let \(\Omega\) be the set of all subsets of \(I\). We define \(T: \Omega \to \Omega\) as follows:

\[
T(D) = \{ w^* | w^* \in I \text{ and for each } w \in I \text{ there exists a } w' \in D \}
\]

such that \(V(w^*, w') \geq V(w, w')\).

Our results are contained in several propositions concerning \(T\).

**Proposition 1 (no regrets).** \(T^N I\) are nested sets:

\(I \supseteq T^1 I \supseteq T^2 I \supseteq \ldots\).

**Proof.** By definition \(TI \subseteq I\). Let \(N\) be such that \(I \supseteq T^1 I \supseteq \ldots \supseteq T^N I\) but \(T^N I \not\subseteq T^{N+1} I\). Let \(\tilde{w} \in (T^{N+1} I \setminus T^N I)\). As \(\tilde{w} \not\in T^N I\) there exists \(\tilde{w} \in I\) such that

\(V(\tilde{w}, w') < V(\tilde{w}, w')\) for all \(w' \in T^{N-1} I\).

As \(T^N [0, 1] \subseteq T^{N-1} I\),

\(V(\tilde{w}, w') < V(\tilde{w}, w')\) for all \(w' \in T^N I\).

Therefore, \(\tilde{w} \not\in T^{N+1} I\), which contradicts the assumption on \(\tilde{w}\). Q.E.D.

**Remark 1.** Note that the proof of Proposition 1 does not depend upon the properties of \(V\).
Proposition 2. If \( D \subseteq I \) and \( D \) is compact then \( TD \) is compact.

Proof. As \( TD \subseteq I \), and therefore is bounded, all we need to show is that \( TD \) is closed. Let \( \tilde{w} \) belong to the closure of \( TD \). For a given \( w \in I \) let \( \{ w_N, w'_N \} \) satisfy \( V(w_N, w'_N) \geq V(w, w'_N) \), where \( w_N \in TD \), \( w'_N \in D \) and \( \lim_{N \to \infty} w_N = \tilde{w} \). As \( D \) is bounded \( w'_N \) has a convergent subsequence, \( w'_J \). Let \( \lim_{J \to \infty} w'_J = \tilde{w} \). As \( D \) is compact, \( \tilde{w} \in D \). Then \( V(\tilde{w}, \tilde{w}) \geq V(w, \tilde{w}) \) by the continuity of \( V \). Therefore \( \tilde{w} \in TD \), so \( TD \) is closed. Q.E.D.

Remark 2. Note that the proof of Proposition 2 depends upon the continuity of \( V \) but not upon its concavity.

Proposition 3 (well posed). If \( D \subseteq I \) and \( D \neq \emptyset \) then \( TD \neq \emptyset \).

Proof. For each \( w' \in D \) \( \exists w^*(w') \in I \) such that \( V(w^*(w'), w') = \max_{w \in D} V(w, w') \) by the compactness of \( I \) and the continuity of \( V \). Then \( w^*(w') \in TD \) for all \( w' \in D \) by the fact that \( V(w^*(w'), w') \geq V(w, w') \) for all \( w \in I \). Q.E.D.

Remark 3. Note that the proof of Proposition 3 depends upon the continuity of \( V \), but not upon its concavity.

We are now ready for our central object of study, \( D^* = \cap_{N=1}^{\infty} T^N I \). \( D^* \) is our proposed alternative to the perfect foresight fixed point.

As \( \{ T^N I \} \) is nested, compact, and non-empty we have:

Corollary 1. \( D^* \) is a non-empty compact set.

Proposition 4 (equilibrium). \( D^* \) is a fixed point of \( T \): \( TD^* = D^* \).

Proof. We consider, and reject, two cases: (1) \( \exists w \in D^* \) such that \( w \notin TD^* \), and (2) \( \exists w \in TD^* \) such that \( w \notin D^* \).

Case 1. \( \tilde{w} \in D^* \), \( \tilde{w} \notin TD^* \). It follows that there exists \( \bar{w} \in I \) such that

\[
V(\tilde{w}, w') < V(\bar{w}, w') \quad \text{for all} \quad w' \in D^*.
\]

But

\[
V(\tilde{w}, w'_N) \geq V(\bar{w}, w'_N) \quad \text{for some} \quad w'_N \in T^N I \quad \text{for all} \quad N.
\]

Let \( w'_J \) be convergent subsequence of \( w'_N \). Let \( \lim_{J \to \infty} w'_J = \tilde{w} \). Then \( V(\tilde{w}, w) \geq V(\bar{w}, \tilde{w}) \) by the continuity of \( V \). Therefore, \( \tilde{w} \notin D^* \). As \( T^N I \)
are nested, for every \( N w'_j \in T^N_i \) for \( J \geq N \). As \( T^N_i \) is compact, \( \lim_{J \to \infty} w'_j = \hat{w} \in T^N_i \). Therefore \( \hat{w} \in D^* = \bigcap_{N-1}^{\infty} T^N_i \), which contradicts \( \hat{w} \notin D^* \).

**Case 2.** \( \hat{w} \in TD^* \), \( \hat{w} \notin D^* \). It follows that there exists \( w \in I \) and an \( N \) such that

\[
V(\hat{w}, w') < V(w, w') \quad \text{for all} \quad w' \in T^N_i.
\]

But

\[
V(\hat{w}, w'') \geq V(w, w'') \quad \text{for some} \quad w'' \in D^*.
\]

As \( D^* \subseteq T^N_i \) this is impossible. Q.E.D.

**Remark 4.** Note that the proof of Proposition 4 depends upon the continuity of \( V \) but not upon its concavity. So far in characterizing \( D^* \) we have not used the concavity of \( V \).

**Proposition 5.** \( D^* \) is an interval (possibly consisting of a point).

**Proof.** We will prove that \( TD \) is an interval for all \( D \subseteq I \). Suppose \( \hat{w} \in (I \setminus TD) \). Then \( \exists w \in I \) such that \( V(\hat{w}, w') < V(w, w') \) for all \( w' \in D \). Suppose \( w > \hat{w} \). Then for all \( \hat{w} \leq \tilde{w} \), \( V(\tilde{w}, w') > V(\hat{w}, w') \) for all \( w' \in D \) by the quasi-concavity of \( V \). Therefore, \( \hat{w} \in (I \setminus TD) \). Similarly if \( \tilde{w} < \hat{w} \), \( \hat{w} \in (I \setminus TD) \). In other words, for all \( \hat{w} \in (I \setminus TD) \), \( TD \) is entirely ‘above’ or ‘below’ \( \hat{w} \), so \( TD \) is an interval. Q.E.D.

**Remark 5.** Note that the proof of Proposition 5 depends upon the quasi-concavity of \( V \) but not upon its strict concavity.

**Proposition 6.** If \( \{ w_i \} \) is a rational expectations (here perfect foresight) equilibrium then \( w_i \in D^* \) for all \( i \).

**Proof.** \( \{ w_i \} \) is a rational expectations equilibrium means that for all \( i \), \( w_i \in I \) and \( V(w_i, w_{i+1}) = \max_{w \in I} V(w, w_{i+1}) \). Clearly \( w_i \in I \) and \( w_i \in TI \) for all \( i \). Clearly if \( w_i \in T^{NI} \) for all \( i \) then \( w_i \in T^{N+1}I \) for all \( i \). Therefore, \( w_i \in \bigcap_{N-1}^{\infty} T^{NI} = D^* \) for all \( i \). Q.E.D.

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1 Indeed note that \( T \), and therefore \( D^* \), is unaffected by any strictly increasing transformation of \( V \).

2 Note that if \( \{ w_i \} \) is a rational expectations equilibrium, so is \( \{ w'_i \ | \ w'_i = w_{i+k} \} \) for every integer \( k \).
Remark 6. Note that the proof of Proposition 6 does not depend upon the properties of $V$, although a rational expectations equilibrium need not exist in general.

Proposition 7. There exists at least one stationary rational expectations equilibrium.

Proof. A stationary rational expectations equilibrium $w^*$ satisfies $w^* \in I$ and $V(w^*, w^*) = \max_{w \in I} V(w, w^*)$. Let $f(w) = \{ x | x \in I \text{ and } V(x, w) = \max_{y \in I} V(y, w) \}$ for $w \in I$. By the continuity and strict quasi-concavity of $V$, $f$ exists and is a continuous function on $I$. By the Brower fixed point theorem $\exists w \in I$ such that $f(w) = w$. $w$ is a stationary rational expectations equilibrium. Q.E.D.

Remark 7. Note that the proof of Proposition 7 depends upon the continuity and strict quasi-concavity of $V$.

Notice that Propositions 5–6 imply that if there are multiple rational expectations equilibria then $D^*$ includes at least them and the entire interval between them. One might think that having isolated multiple rational expectations equilibria is qualitatively different from having a continuum of them. As $D^*$ is a continuum in either case, this analysis suggests otherwise. Moreover, in some applications in which there is a unique stationary rational expectations equilibrium, $w^*$, $D^* = w^*$, and in some $D^* \neq w^*$ (although $w^* \subset D^*$).

It may be worth noting that our example is easily reinterpreted as a symmetric static game. Suppose there is a continuum of individuals $s \in I$, who taken together make a decision function $w(s)$. Let $w - w(s)$ and $w' = \int w(s) \, ds$ be the arguments of $V$ for each individual $s$. Given the proof of Proposition 5, we do not have an aggregation problem. As $w'$ is unaffected by any single choice $w(s)$, we can assume independence as argued in Bryant (1982). $D^*$ is the contraction by dominance solution, and a stationary rational expectations equilibrium $w^*$ is a Nash equilibrium. Also, our original example can be run backwards in time, but with previous decisions unobservable.

For example, consider $V(w, w') = -w^2 - (w')^2$ and $V(w, w') = -(2w - 1)(2w' - 1))^2$, respectively.
References

Pearce, David G., 1983, Rationalizable strategic behavior and the problem of perfection (Princeton University, Princeton, NJ)